# Formal group laws and informal group disobedience II
# GRK Retreat 2024

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# Outline

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- There is a universal formal group law.
- There is a universal complex-oriented ring spectrum.
- Every complex-oriented ring spectrum determines a formal group law.
- A formal group law creates a spectrum under a certain condition (if time permits).

# Universal formal group law

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- $c_{i,0} = c_{0,i} = 0$ if $i \neq 1$, and $c_{1,0} = c_{0,1} = 1$,
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### Definition

The *Lazard ring* is $L := \mathbf{Z}[c_{ij}]/Q$, where $Q$ is the ideal generated by the these polynomials.

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A useful description of $L$:

## Theorem (Lazard)

*There is an isomorphism $L \cong \mathbf{Z}[t_1, t_2, \ldots]$.*

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In particular, to write down a formal group law over a commutative ring $R$, one just needs to select a countable sequence of elements of $R$. In particular, formal group laws exist in abundance.

**Definition**

A *spectrum* is a sequence of pointed spaces

$$E := (E_0, E_1, \dots)$$

equipped with an equivalence $E_i \simeq \Omega E_{i+1}$ for every $i$, where $\Omega := \mathrm{Map}(S^1, -)$.

# Complex-oriented cohomology theory

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Every spectrum $E$ yields a homology theory

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$$E_i(X) := [S^i, X \wedge E]$$

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for space $X$. We set $E_* := E_*(\mathrm{pt})$ and $E^* := E^*(\mathrm{pt})$. Then $E_* \cong E^*$ with reverse grading.

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Let $\bar{t}$ be the element of $E^2(\mathbf{CP}^1)$ corresponding the unit of $E^0(\mathrm{pt})$.
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## Definition

A *complex-orientation* on a homotopy commutative ring spectrum $E$ is an element $t \in E^2(\mathbf{CP}^\infty)$ that maps to $\overline{t} \in E^2(\mathbf{CP}^1)$.

### Example

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### Example

The element $t := [\mathcal{O}(-1)] - 1 \in K^0(\mathbf{CP}^\infty) \cong K^2(\mathbf{CP}^\infty)$ is a complex-orientation on the complex $K$-theory spectrum $KU$, where $\mathcal{O}(-1)$ is the tautological complex line bundle over $\mathbf{CP}^\infty$.

**Theorem**

*Let $E$ be a complex-oriented ring spectrum. For a space $X$, we have*

$$E^*(X \times \mathbf{CP}^\infty) \cong E^*(X) \otimes \mathbf{Z}[[t]].$$

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**Proof.**

Similar to the computation of $H^*(\mathbf{CP}^\infty; \mathbf{Z})$. □

# Complex-oriented cohomology theory

Using $\mathbf{CP}^\infty \cong BS^1$, we have the multiplication $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$, which induces

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Together with the above theorem, we obtain

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Let $f(x, y)$ denote the image of $t$ under this map.

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## Slogan

Every complex-oriented ring spectrum determines a formal group law.

**Definition**

The *Thom space* of a rank $n$ complex vector bundle $\mathcal{E} \to X$ (equipped with a metric) is

$$\mathrm{Th}(\mathcal{E}) := \mathrm{D}(\mathcal{E})/\mathrm{S}(\mathcal{E}),$$

where $\mathrm{D}(\mathcal{E})$ and $\mathrm{S}(\mathcal{E})$ are the unit disk bundle and unit sphere bundle.

**Theorem (Thom isomorphism)**

*For a complex-oriented ring spectrum $E$, we have*

$$E^*(X) \cong E^{*+2n}(\mathrm{Th}(\mathcal{E})).$$

# Complex bordism

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**Proof.**

Use $\mathrm{Th}(\mathcal{E}) \simeq \mathbf{P}(\mathcal{E} \oplus \mathcal{O})/\mathbf{P}(\mathcal{E})$ and the projective bundle formula. □

# Complex bordism

**Definition**

For a topological group $G$, the *classifying space of $G$* is the geometric realization of the simplicial space

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G \times G \substack{\rightarrow\\\rightarrow\\\rightarrow} G \rightrightarrows \mathrm{pt}.$$

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**Definition**

For a pointed space $X$, its *infinite suspension* $\Sigma^\infty X$ is the spectrum associated with $(X, S^1 \wedge X, S^2 \wedge X, \ldots)$.

# Complex bordism

**Definition**

Consider the classifying space $BU(n)$ of the unitary group $U(n)$ and its tautological bundle $\mathcal{T}_n$.
The *bordism spectrum* is $MU := \operatorname{colim} MU(n)$, where $MU(n) := \Omega^{2n}\Sigma^{\infty}\mathrm{Th}(\mathcal{T}_n)$.

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For the ring structure, construct $MU(m) \wedge MU(n) \to MU(m+n)$.
The canonical map $MU(1) \to MU$ yields a class
$t \in MU^2(\mathrm{Th}(\mathcal{T}_1)) \cong MU^2(\mathbf{P}^{\infty}/\mathrm{pt})$. □

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**Theorem**

*Let $E$ be a homotopy commutative ring spectrum. Then there is a one-to-one correspondence between complex-orientations on $E$ and ring maps $MU \to E$.*

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Show that these two constructions are inverses to each other. □

**Slogan**

$MU$ is the universal complex-oriented ring spectrum.

# Comparison of two universalities

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**Proof.**

Even though the theorem is conceptual, the proof is computational.
Step 1. Show that $L \wedge H\mathbf{Q} \to MU^* \wedge H\mathbf{Q}$ is an isomorphism by computing $H^*(MU, \mathbf{Z})$.

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**Proof.**

Step 2. The cosimplicial diagram

$$MU \wedge H\mathbf{F}_p \rightrightarrows MU \wedge H\mathbf{F}_p \wedge H\mathbf{F}_p \overrightarrow{\rightrightarrows} \cdots$$

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yields the Adams spectral sequence

$$E_1^{ij} = H_i(MU; \mathbf{F}_p) \otimes_{\mathbf{F}_p} (\mathcal{A}^\vee)^{\otimes j} \Rightarrow (\pi_{i+j} MU)_p^\wedge,$$

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where $\mathcal{A}^\vee := \pi_*(H\mathbf{F}_p \otimes H\mathbf{F}_p)$ denotes the dual Steenrod algebra.

# Comparison of two universalities

**Theorem (Quillen)**

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**Proof.**

Step 3. Show $E_2^{**} \cong \mathbf{F}_p[c_0, c_1, \ldots]$, where $c_i$ has total degree $2i$. In particular, the Adams spectral sequence degenerates at the second page by the degree consideration.

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Step 4. Show $(\pi_* MU)_p^\wedge \cong \mathbf{Z}_p[u_1, u_2, \dots]$.
Step 5. Analysis the induced map $L_p^\wedge \cong \mathbf{Z}_p[t_1, t_2, \dots] \to \mathbf{Z}_p[u_1, u_2, \dots]$, and show that this is indeed an isomorphism. □

# Landweber exact functor theorem

For a prime $p$, let $v_n \in L$ be the coefficient of $t^{p^n}$ in the $p$-series $[p](t)$, where $[0](t) = 0$ and $[m](t) = \ell([m-1](t), t)$ for $m \geq 1$.

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### Theorem (Landweber exact functor theorem)

*Let $M$ be a graded $L$-module. If the sequence $p, v_1, v_2, \ldots$ is $M$-regular every prime $p$, then there exists a spectrum $E$ such that*

$$E_*(X) \cong MU_*(X) \otimes_L M$$

*for every space $X$.*

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is exact for CW pair $(X, A)$ and integer $n$.

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If $M$ were flat over $L$, then there would be no problem. However, $L$ is an infinite polynomial ring, and $M$ is usually not flat over $L$.

Proof.

The crucial idea is to consider the *moduli stack of formal groups* $\mathfrak{M}_{\mathrm{FG}} := \mathrm{Spec}(L)/G^+$ with

$$G^+(R) := \{g \in R[[x]] : g(x) = b_1 x + b_2 x^+ \cdots , b_1 \in R^\times\}$$

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A graded commutative ring $R$ with a formal group law is an $L$-algebra and hence an $L$-module.

**Slogan**

We can create $E$ from a formal group law under a certain condition.

### Example

Consider $\mathbf{Z}$ with the formal group law $f(x, y) := x + y$. Then $[p](t) = pt$, so $v_n = 0$ for $n \geq 1$. Hence $p, v_1$ is not $\mathbf{Z}$-regular. We cannot apply the Landweber exact functor theorem to this example.

# Landweber exact functor theorem

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## Example

We get singular homology with $\mathbf{Q}$-coefficients from the Landweber exact functor theorem:
It is known that

$$H_*(X; \mathbf{Q}) \cong MU_*(X) \otimes_L \mathbf{Q}.$$

## Example

We have the formal group law over $\mathbf{Z}[\beta, \beta^{-1}]$ with $|\beta| = -2$ given by

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We get K-theory from the Landweber exact functor theorem: Conner and Floyd proved that there is an isomorphism

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## Example

We get the *Brown-Peterson specturm BP* from the Landweber exact functor theorem with

$$M := \mathbf{Z}_{(p)}[t_1, t_2, \ldots]/(t_i)_{i+1 \neq p^k}.$$